

International Journal of Geometric Methods in Modern Physics  
© World Scientific Publishing Company

## ALTERNATIVE ALGEBRAIC STRUCTURES FROM BI-HAMILTONIAN QUANTUM SYSTEMS

G. MARMO\*

*Dip. Sc. Fisiche and INFN-Napoli, Università Federico II, Via Cintia  
Napoli, 80126, Italy  
marmo@na.infn.it*

G. SCLARICI

*Dip. Fisica and INFN-Lecce, Università di Lecce, Via per Arnesano  
Lecce, 73100, Italy  
sclarici@le.infn.it*

A. SIMONI

*Dip. Sc. Fisiche and INFN-Napoli, Università Federico II, Via Cintia  
Napoli, 80126, Italy  
simoni@na.infn.it*

F. VENTRIGLIA

*Dip. Sc. Fisiche and INFN-Napoli, Università Federico II, Via Cintia  
Napoli, 80126, Italy  
ventriglia@na.infn.it*

Received (Day Month Year)

Revised (Day Month Year)

We discuss the alternative algebraic structures on the manifold of quantum states arising from alternative Hermitian structures associated with quantum bi-Hamiltonian systems. We also consider the consequences at the level of the Heisenberg picture in terms of deformations of the associative product on the space of observables.

*Keywords:* Quantum Mechanics; Geometric Structures; Operator Algebra; Deformation of Associative Algebras; Alternative Hermitian Structures

### 1. Introduction

Complete integrability in the classical framework has been associated with the existence of alternative Hamiltonian descriptions. As a matter of fact, integrability of a given dynamical system admitting alternative Hamiltonian descriptions is usually implied by the compatibility of the two Hamiltonian structures, when they are

\*Dip. Sc. Fisiche, Università Federico II, Via Cintia 80126 Napoli, Italy.

2 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

generic.

Because quantum theory is considered to be more fundamental than the classical one, one usually expects that quantum systems, whose corresponding classical limits are completely integrable, should admit alternative quantum descriptions. Recently, some alternative “quantum structures” have been identified as alternative Hermitian structures on the vector space of physical states in the Schroedinger picture,[1,2,3,4,5,6,7,8] or as alternative associative products on the space of observables in the Heisenberg picture.[9] In some of previous papers the notion of mutually compatible alternative structures has been analyzed to find out how many different dynamical systems may be bi-Hamiltonian with respect to two given structures.

It is usually stated that the Schroedinger picture and the Heisenberg picture are equivalent. In this paper, we would like to consider those alternative structures in the Heisenberg picture which correspond to the alternative Hermitian structures which we find in the Schroedinger picture. More generally, when a dynamical system is identified with a derivation of suitable products, we would like to consider the existence of alternative and mutually compatible algebraic structures on the same carrier space. In some sense, we use the notion of compatibility for Poisson brackets as a guiding idea to define a notion of compatibility for algebraic structures admitting a common derivation.

Moreover, a systematic formulation of quantum mechanics on quaternionic Hilbert spaces exists [10] and, in the last few years, there have been some suggestions that quaternionic quantum mechanics (QQM) may be useful to classify positive maps in standard quantum mechanics.[11] Thus, we introduce the notion of compatible quaternionic quantum bi-Hamiltonian dynamical systems and discuss their Schroedinger and Heisenberg representations.

It is possible a discussion of both the standard and quaternionic formulation of quantum mechanics in a unified conceptual framework arising from a geometrization of quantum mechanics. Roughly speaking, it is possible to start with a real differential manifold  $\mathcal{M}$  as carrier space, instead of the Hilbert space  $\mathcal{H}$ . Then operators acting on  $\mathcal{H}$  may be associated with (1-1)-tensors acting on the tangent space  $T\mathcal{M}$ , and the coefficients of such tensor fields at each point may be in turn real, or complex or even quaternionic numbers. In other words, only the coefficients of (1-1)-tensor fields are complex valued or quaternionic valued functions, while the carrier space  $\mathcal{M}$  remains a real differential manifold.

## 2. Geometrization of Quantum Mechanics

In this section we discuss the relevant geometric structures which appear in standard quantum mechanics and the relations among them, in the framework of our geometrization.[7,12]

To avoid technicalities, for the time being, while we deal with general aspects, we shall confine ourselves to finite dimensional carrier spaces.

We start with a complex Hilbert space  $\mathcal{H}$  and consider its realification  $\mathcal{H}^{\mathbb{R}}$ .

In other words, given a basis  $\{\varphi_k\}$ , any vector  $\psi$  will be replaced by its complex components  $(q_k + ip_k)$  in  $\mathcal{H}$  and with real components  $(q_k, p_k)$  in  $\mathcal{H}^{\mathbb{R}}$ . Now let the real vector space  $\mathcal{H}^{\mathbb{R}}$ , considered as a contractible real manifold  $\mathcal{M}$ , be equipped with a symplectic structure  $\omega$ , a non-degenerate 2-form such that

$$d\omega = 0 \quad (1)$$

Then  $\dim \mathcal{M}$  is even, say  $2n$ . A global Darboux chart  $\{q_k, p_k\}$  endows  $\mathcal{M}$  with a real linear structure  $\Delta$ , the infinitesimal generator of dilation (often also called the Liouville vector field or the Euler operator) whose tensorial expression is provided by

$$\Delta = \sum_k \left( q_k \frac{\partial}{\partial q_k} + p_k \frac{\partial}{\partial p_k} \right). \quad (2)$$

In this chart, we define a linearly admissible complex structure, i.e. a (1-1) tensor field  $J$  commuting with  $\Delta$  such that  $J^2 = -1$ . Then construct a tensor  $g$  as

$$g = \omega \circ J. \quad (3)$$

The triple  $(g, \omega, J)$  is (linearly) admissible if  $g$  results an Euclidean metric tensor in a global Darboux chart. This generalizes the definition of admissible triple  $(g, \omega, J)$  we have given in Ref. [1]. It is also possible to construct a (linearly) admissible triple  $(g, \omega, J)$  starting from  $g$  and  $J$ , following the lines of Ref. [13].

Along with a symplectic structure, an associated Poisson structure  $\Lambda = \omega^{-1}$  may be defined in the chosen global Darboux chart as the contravariant counterpart of  $\omega$ , it corresponds to the standard Poisson Brackets associated with a symplectic structure.

To completely turn entities depending on the vector space structure on the space of states into tensorial objects, we notice that with every matrix  $A \equiv \|A_k^j\| \in \mathfrak{gl}(2n, \mathbb{R})$  acting on  $\mathcal{H}^{\mathbb{R}}$  we can associate both a linear vector field, acting on  $\mathcal{M}$ :

$$X_A : \mathcal{M} \rightarrow T\mathcal{M}, \quad \psi \rightarrow (\psi, A\psi) \quad (4)$$

and a  $(1-1)$ -tensor, acting on  $T\mathcal{M}$ :

$$T_A : T\mathcal{M} \rightarrow T\mathcal{M}, \quad (\psi, \varphi_\psi) \rightarrow (\psi, A\varphi_\psi). \quad (5)$$

So, when  $A = 1$ , we get the linear structure  $\Delta$ :

$$\Delta : \psi \rightarrow (\psi, \psi). \quad (6)$$

This vector field allows to identify  $T_\psi \mathcal{M}$  with  $\mathcal{M}$ , i.e. the base manifold  $\mathcal{M}$  gets a vector space structure from the one on its tangent space at the origin.

Then,  $X_A$  and  $T_A$  are connected by the linear structure  $\Delta$ :

$$T_A(\Delta) = X_A \quad (7)$$

and are both homogeneous of degree zero, i.e.

$$L_\Delta X_A = L_\Delta T_A = 0. \quad (8)$$

4 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

While the correspondence  $A \rightarrow T_A$  is a full associative algebra and a corresponding Lie algebra isomorphism, the correspondence  $A \rightarrow X_A$  is instead only a Lie algebra (anti)isomorphism, that is

$$T_A \circ T_B = T_{AB} \quad (9)$$

while

$$[X_A, X_B] = -X_{[A, B]}. \quad (10)$$

Moreover, for any  $A, B \in \mathfrak{gl}(2n, \mathbb{R})$  :

$$L_{X_A} T_B = -X_{[A, B]}. \quad (11)$$

Equation (11) allows for the definition of constant tensors: a tensor  $T_B$  is constant with respect to the linear structure  $\Delta$  when

$$L_{\Delta} T_B = 0. \quad (12)$$

We recall that an Hermitian tensor  $h$  on  $\mathcal{M}$ , can be defined as a map:

$$h : T_{\psi}\mathcal{M} \times T_{\psi}\mathcal{M} \longrightarrow \mathbb{C}, \quad (13)$$

such that

$$h(\Delta, \Delta) = g(\Delta, \Delta); \quad h(\Delta, J(\Delta)) = i\omega(\Delta, J(\Delta)) \quad (14)$$

are respectively a real valued and a purely imaginary valued quadratic function of real variables.

We shall use the real quadratic function

$$\mathfrak{g} = \frac{1}{2}g(\Delta, \Delta) \quad (15)$$

as the Hamiltonian generating function of the field  $\Gamma$  :

$$i_{\Gamma}\omega = -d\mathfrak{g}. \quad (16)$$

It would have been possible to start with  $J$  and  $g$  to recover  $\omega$  by means of the exterior derivative associated with  $J$ . [14] Indeed, with the help of

$$d_J = d \circ J - J \circ d, \quad (17)$$

the symplectic structure is recovered through

$$dd_J \left( \frac{1}{2}g(\Delta, \Delta) \right) = \omega. \quad (18)$$

It is possible to show that

$$\Gamma = J(\Delta) \quad (19)$$

and  $J(\Gamma) = -\Delta$ . The vector field  $\Gamma$  preserves all three structures  $g, \omega$  and  $J$ . Thus the vector field  $\Gamma$  will be a generator of a one-parameter group of unitary

transformations and may be associated with a Schroedinger-type equation (we set  $\hbar = 1$ )

$$J \frac{d}{dt} \psi = H \psi, \quad (20)$$

where  $H\psi$  is the second component of  $\Gamma(\psi)$ . The dynamics is therefore determined by the vector field  $\Gamma$ .

To search for alternative descriptions, we look for all Hermitian tensors on  $\mathcal{M}$  invariant under the dynamical evolution. We have to consider the equation  $L_\Gamma h = 0$  for the unknown  $h$ ,  $h$  representing an unknown Hermitian tensor on  $\mathcal{M}$ . This is equivalent to  $L_\Gamma \omega = 0$ ,  $L_\Gamma g = 0$ ,  $L_\Gamma J = 0$ , so that we may solve for  $L_\Gamma h = 0$  by starting solving for  $L_\Gamma \omega = 0$ . In this way we take into account, as discussed in Ref. [15], that both  $\omega$  and  $g$  may be point-dependent. Here, rather than dealing with the general theory, we limit ourselves to discuss a simple example.

### 2.1. A simple example

For a one-dimensional system, in a global Darboux chart  $(q, p)$  of  $\mathcal{M} = \mathbb{R}^2$ , we consider the dynamics described by the vector field

$$\Gamma = p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p} \quad (21)$$

with standard Hamiltonian description provided by

$$\omega_s = dq \wedge dp, \quad H_s = \frac{1}{2}(q^2 + p^2). \quad (22)$$

The other relevant tensors are

$$g = dq \otimes dq + dp \otimes dp, \quad (23)$$

$$J = dp \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial p}, \quad (24)$$

$$\Delta = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p} \quad (25)$$

and

$$\Lambda = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}. \quad (26)$$

The most general symplectic structure solving the previously stated equation for  $\omega$  is given by [16]

$$\omega_F = F(H_s) dq \wedge dp \quad (27)$$

with  $F(H_s)$  vanishing nowhere.

By performing simple computations, one finds for every  $\omega_F$  a Darboux chart

$$P = p(1 + f(H_s)), \quad Q = q(1 + f(H_s)), \quad (28)$$

6 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

where  $f$  is any solution of the differential equation

$$\frac{d}{ds}[s(1+f(s))^2] = F(s). \quad (29)$$

As a particular case, we may consider a nonlinear diffeomorphism of the form

$$P_\lambda = p(1 + \lambda H_s), \quad Q_\lambda = q(1 + \lambda H_s), \quad (30)$$

where a real parameter  $\lambda$  appears. From Eq.(30) we get

$$dP_\lambda = (1 + \lambda(q^2 + 3p^2))dp + 2\lambda pq dq, \quad dQ_\lambda = (1 + \lambda(p^2 + 3q^2))dq + 2\lambda pq dp. \quad (31)$$

By using Eq.(31), the metric tensor  $g_\lambda$  and the symplectic form  $\omega_\lambda$  can be respectively obtained as functions of  $q, p$  as

$$\begin{aligned} g_\lambda &= dP_\lambda \otimes dP_\lambda + dQ_\lambda \otimes dQ_\lambda \\ &= [(1 + \lambda(q^2 + 3p^2))^2 + 4\lambda^2 p^2 q^2] dp \otimes dp + \\ &\quad [(1 + \lambda(p^2 + 3q^2))^2 + 4\lambda^2 p^2 q^2] dq \otimes dq + \\ &\quad 4\lambda pq(1 + 2\lambda(q^2 + p^2))[dq \otimes dp + dp \otimes dq] \end{aligned} \quad (32)$$

and

$$\begin{aligned} \omega_\lambda &= dP_\lambda \wedge dQ_\lambda \\ &= [(1 + \lambda(q^2 + 3p^2))^2 - 4\lambda^2 p^2 q^2] dp \wedge dq. \end{aligned} \quad (33)$$

The associated Poisson bracket in the  $(Q, P)$  coordinates will be given by the following expression

$$\{Q, P\}_\lambda = \frac{1}{[(1 + \lambda(q^2 + 3p^2))^2 - 4\lambda^2 p^2 q^2]} \{q, p\} \quad (34)$$

due to the use of a non-canonical transformation. As for the complex structure we have

$$J(p, q) = dp \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial p}.$$

In the  $(Q, P)$  coordinates the dynamical vector field  $\Gamma$  has the form

$$\Gamma = P \frac{\partial}{\partial Q} - Q \frac{\partial}{\partial P}. \quad (36)$$

We see immediately that  $\Gamma$  is also Hamiltonian with respect to an alternative Poisson bracket given by  $\{Q, P\} = 1$ , along with the complex structure

$$J(P, Q) = dP \otimes \frac{\partial}{\partial Q} - dQ \otimes \frac{\partial}{\partial P}. \quad (37)$$

We remark that the two vector fields

$$\Delta(p, q) = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p}, \quad \Delta(P, Q) = Q \frac{\partial}{\partial Q} + P \frac{\partial}{\partial P} \quad (38)$$

define two alternative linear structures on  $\mathcal{M}$ , which are not linearly related. Indeed the following tensor  $T$  :

$$T = \frac{P}{p} \frac{\partial}{\partial P} \otimes dp + \frac{Q}{q} \frac{\partial}{\partial Q} \otimes dq, \quad (39)$$

written for simplicity in mixed coordinates, maps one linear structure into the other:

$$T(\Delta(p, q)) = T\left(q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p}\right) = Q \frac{\partial}{\partial Q} + P \frac{\partial}{\partial P} = \Delta(P, Q). \quad (40)$$

The existence of these two alternative linear structures means that we may compose (add) solutions for  $\Gamma$  in alternative ways to get new solutions. The fact that these linear structures are not linearly related means that the image of the composition (sum) is not the composition (sum) of the images.[17]

### 3. Alternative compatible Hermitian structures

In general, two symplectic structures associated with a classical bi-Hamiltonian system do not admit a common Darboux chart. When they are “constant” in the same global Darboux chart, one can find a linear transformation (that is, a diffeomorphism commuting with  $\Delta$ ) which connects the two symplectic structures.

In this section we review some results on quantum bi-Hamiltonian systems concerning tensor fields which are compatible with (i.e. constant with respect to) a given linear structure  $\Delta$  and compatible with (i.e. commuting with) an assigned complex structure  $J$ .

Suppose that two admissible triples  $(g_1, J_1, \omega_1)$  and  $(g_2, J_2, \omega_2)$  are given on  $\mathcal{M} = \mathcal{H}^{\mathbb{R}}$ . Then, by complexification, we get two different Hilbert spaces, each one with its proper multiplication by complex numbers and with its proper Hermitian structure. Quantum theory in the usual Schroedinger formulation, when we start from a given Schroedinger dynamics, leads quite naturally to consider identical complex structures in the two triples and the condition  $J_1 = J_2$  is a sufficient condition for compatibility [1]. On the contrary, in the real context it is possible to consider the case of two admissible triples with  $J_1 \neq J_2$  which are compatible [8].

Then, we may assume compatibility, by taking two Hermitian structures,  $h_1(., .)$  and  $h_2(., .)$ , on the same complex Hilbert space  $\mathcal{H}$ , coming from two admissible triples admitting  $J_1 = J_2$ . We search for the group of automorphism which leave both  $h_1$  and  $h_2$  invariant, that is the bi-unitary transformations group.

By using the Riesz’s theorem a bounded, positive operator  $G$  may be defined, which is self-adjoint both with respect to  $h_1$  and  $h_2$ , as:

$$h_2(x, y) = h_1(Gx, y), \quad \forall x, y \in \mathcal{H}. \quad (1)$$

Moreover, any bi-unitary transformation  $U$  must commute with  $G$ . Indeed:

$$h_1(x, U^\dagger G U y) = h_1(Ux, G U y) = h_2(Ux, U y) = h_2(x, y) = h_1(Gx, y) = h_1(x, G y)$$

8 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

and from this

$$U^\dagger G U = G \Leftrightarrow [G, U] = 0. \quad (3)$$

Therefore the group of bi-unitary transformations is contained in the commutant  $G'$  of the operator  $G$ .

To visualize these transformations, let us consider the bi-unitary group of transformations when  $\mathcal{H}$  is finite-dimensional. In this case  $G$  is diagonalizable and the two Hermitian structures result proportional in each eigenspace of  $G$  *via* the eigenvalue. Then the group of bi-unitary transformations is given by

$$U(d_1) \times U(d_2) \times \dots \times U(d_m), \quad d_1 + d_2 + \dots + d_m = n = \dim \mathcal{H}, \quad (4)$$

where  $d_k$  denotes the degeneracy of the  $k$ -th eigenvalue of  $G$ .

Each Hermitian structure on  $\mathcal{H}$  defines a different realization of the unitary group as a group of transformations. The intersection of these two groups identifies the group of bi-unitary transformations.

Now we can further qualify and strengthen the compatibility condition by stating the following [1]:

**Definition 1** *Two Hermitian forms are said to be in generic relative position when the eigenvalues of  $G$  are non-degenerate.*

Then, if  $h_1$  and  $h_2$  are in generic position, the group of bi-unitary transformations becomes

$$\underbrace{U(1) \times U(1) \times \dots \times U(1)}_{n \text{ factors}}.$$

In other words, this means that  $G$  generates a complete set of commuting observables.

Moreover, the following proposition holds [8]:

**Proposition 1** *Two Hermitian forms are in generic relative position if and only if their connecting operator  $G$  is cyclic.*

This shows that definition (1) may be equivalently formulated as:

**Definition 2** *Two Hermitian forms are said to be in generic relative position when their connecting operator  $G$  is cyclic.*

The genericity condition can also be restated in a purely algebraic form as follows [1]:

**Definition 3** *Two Hermitian forms are said to be in generic relative position when  $G'' = G'$ , i.e. when the bi-commutant of  $G$  coincides with the commutant of  $G$ .*

Equivalence of definitions (1), (2), and (3) is apparent and hold in the finite as well as in the infinite-dimensional case [8].

**Remark** In a more abstract setting, bi-Hamiltonian dynamical systems associated with systems 1 and 2 can be seen as the infinitesimal generators of one-parameter groups in the intersection

$$\text{Aut } S_1 \cap \text{Aut } S_2,$$



where  $\text{Aut } S$  denotes the automorphisms of the structure  $S$ . In classical and in quantum mechanics,  $S_1$  and  $S_2$  will represent symplectic and Hermitian structures respectively associated with the systems 1 and 2 and the intersection  $\text{Aut } S_1 \cap \text{Aut } S_2$  will be associated with all bi-Hamiltonian dynamical systems. In this geometrical context the compatibility can be restated as follows:

**Definition 4** *Two structures  $S_1$  and  $S_2$  will be said compatible iff the intersection  $\text{Aut } S_1 \cap \text{Aut } S_2$  is non void and non trivial.*

Moreover, the following definition qualify and strengthen the compatibility condition:

**Definition 5** *Two (compatible) structures  $S_1$  and  $S_2$  will be said to be in relative generic position iff the intersection  $\text{Aut } S_1 \cap \text{Aut } S_2$  is minimal and non trivial.*

Equivalence of definitions (1), (2), (3) and (5) is apparent.

#### 4. Alternative compatible quaternionic Hermitian structures

Equations of motion in (right) quaternionic Hilbert space  $\mathcal{H}^{\mathbb{Q}}$  are defined by the Schroedinger equation (we set  $\hbar = 1$ ) [10]:

$$\frac{d}{dt}\psi = -\tilde{H}\psi. \quad (1)$$

The dynamics is determined by the linear operator  $\tilde{H}$ . To search for alternative Hermitian quaternionic descriptions, we look for all scalar products on  $\mathcal{H}^{\mathbb{Q}}$  invariant under the dynamical evolution.

Along the lines of previous sections, if we define  $\Gamma : \mathcal{H}^{\mathbb{Q}} \rightarrow T\mathcal{H}^{\mathbb{Q}}$  to be the map  $\psi \rightarrow (\psi, -\tilde{H}\psi)$ , we have to solve for  $L_{\Gamma}h = 0$ , where now  $h$  is an unknown Hermitian quaternionic structure on  $\mathcal{H}^{\mathbb{Q}}$ .

We recall [18] that any  $h$  on  $\mathcal{H}^{\mathbb{Q}}$  defines an Euclidean metric  $g$ , three symplectic forms  $\omega_a$  ( $a = 1, 2, 3$ ) and three complex structures  $J_a$  satisfying the quaternionic algebra on the realification  $\mathcal{H}^{\mathbb{R}}$  of the right quaternionic Hilbert space  $\mathcal{H}^{\mathbb{Q}}$ :

$$h(.,.) =: g(.,.) + ig(J_1.,.) + jg(J_2.,.) + kg(J_3.,.) \quad (2)$$

where  $i, j, k$  are the quaternion units satisfying  $ij = -ji = k$ ,  $i^2 = j^2 = k^2 = -1$ .

The imaginary parts of  $h$  are symplectic structures  $\omega_a$  on the real vector space  $\mathcal{H}^{\mathbb{R}}$ :

$$\omega_a(.,.) := g(J_a.,.). \quad (3)$$

In the quaternionic case we define *admissible* triple by  $(g, \mathbf{J}, \omega)$ , where  $\mathbf{J} = (J_1, J_2, J_3)$  and  $\omega = (\omega_1, \omega_2, \omega_3)$  define hypercomplex and hypersymplectic structures respectively.[19]

Introducing now two right quaternionic Hermitian structures  $h_1$  and  $h_2$  on the real space  $\mathcal{H}^{\mathbb{R}}$ , coming from two admissible triples  $(g_1, \mathbf{J}_1, \omega_1)$  and  $(g_2, \mathbf{J}_2, \omega_2)$ , we will show that sufficient condition for compatibility according with definition 4, is that the hypercomplex structures  $\mathbf{J}_1$  and  $\mathbf{J}_2$  are the same, up to a transformation

10 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

of a right  $SU(2)$  group, i. e.  $h_1$  and  $h_2$  are defined, up to an automorphism of  $\mathbb{Q}$ , on the same right quaternionic Hilbert space  $\mathcal{H}^{\mathbb{Q}}$ .

To show this, we start resuming well known results about symmetry transformations on right quaternionic Hilbert spaces.

Physical states in QQM are in one-to-one correspondence with unit rays of the form  $|\psi\rangle = \{|\psi\rangle\theta\}$ , with  $|\psi\rangle$  a unit normalized vector and  $\theta$  a “quaternionic phase” of magnitude unity. A symmetry operation  $\mathcal{S}$  of the system is a mapping of the unit rays  $|\psi\rangle$  into images  $|\psi'\rangle$ , which preserves all transition probabilities:

$$\mathcal{S}|\psi\rangle = |\psi'\rangle \quad (4)$$

$$|\langle\psi'|\varphi'\rangle| = |\langle\psi|\varphi\rangle|.$$

In CQM case, the classical Wigner theorem states that the unit ray mapping of the previous equation can be replaced, by an appropriate choice of ray representatives, by a mapping  $\mathcal{U}|\psi\rangle = |\psi'\rangle$  acting on the vectors  $|\psi\rangle$  of Hilbert space, with  $\mathcal{U}$  either unitary or antiunitary. This theorem was generalized by Bargmann to the case of QQM [20]. The generalized theorem states that for a quantum mechanics based on a field  $\mathbb{F}$ , the unit ray mapping of the previous equation can always be replaced by a vector mapping  $\mathcal{U}|\psi\rangle = |\psi'\rangle$  where  $\mathcal{U}$  denotes an additive projective unitary transformation obeying

$$\mathcal{U}(|\psi\rangle + |\varphi\rangle) = \mathcal{U}|\psi\rangle + \mathcal{U}|\varphi\rangle \quad (6)$$

$$\mathcal{U}|\psi q\rangle = \mathcal{U}|\psi\rangle \text{Aut}_{\mathcal{U}}(q)$$

$$h(\mathcal{U}\psi, \mathcal{U}\varphi) = \text{Aut}_{\mathcal{U}}(h(\psi, \varphi)),$$

with  $\text{Aut}_{\mathcal{U}}(q)$  a  $\mathcal{U}$ -dependent automorphism of the field  $\mathbb{F}$ . When  $\mathbb{F}$  is the field of quaternions  $\mathbb{Q}$ , the automorphism  $\text{Aut}_{\mathcal{U}}$  must have the form

$$\text{Aut}_{\mathcal{U}}(q) = \bar{\theta}_{\mathcal{U}} q \theta_{\mathcal{U}}, \quad |\theta_{\mathcal{U}}| = 1$$

where  $\bar{\theta}$  denotes the quaternion conjugate of  $\theta$ .

Defining now a new operator  $U$  by

$$U|\psi\rangle = \mathcal{U}|\psi\rangle \bar{\theta}_{\mathcal{U}}$$

for arbitrary  $|\psi\rangle$  we immediately obtain from Eq.(6) that

$$U|\psi q\rangle = U|\psi\rangle q$$

$$h(U\psi, U\varphi) = h(\psi, \varphi)$$

and so  $U$  gives a quaternion linear, unitary vector mapping. Then, in QQM, the unit ray mapping of Eq. (4) can always be replaced by a unitary mapping  $U$  on the same Hilbert space. With this fact in mind, we now search for the unitary transformations which leave both  $h_1$  and  $h_2$  invariant, that is the bi-unitary transformations group. This group of transformations allows us to construct the group of automorphisms of both  $h_1$  and  $h_2$ .

By using the Riesz's theorem (that also holds for right quaternionic Hilbert spaces [21]) a bounded, positive operator  $G$  may be defined, which is self-adjoint both with respect to  $h_1$  and  $h_2$ , as:

$$h_2(x, y) = h_1(Gx, y), \quad \forall x, y \in \mathcal{H}^{\mathbb{Q}}. \quad (9)$$

Moreover, as in the case of complex Hilbert space, any bi-unitary transformation  $U$  must commute with  $G$ :

$$U^\dagger G U = G \Leftrightarrow [G, U] = 0. \quad (10)$$

Therefore the quaternionic group of bi-unitary transformations is contained in the commutant  $G'$  of the operator  $G$ . In the case that  $G$  admits discrete spectrum, its spectral decomposition reads [10]

$$G = \sum_m \sum_{a=1}^{d_m} |u_m, a\rangle \lambda_m \langle u_m, a|, \quad \lambda_m > 0,$$

where  $\{|u_m, a\rangle\}$  is the eigenvectors basis of  $G$  (orthonormal with respect to both the Hermitian structures) and  $a$  is a degeneracy label. The quaternionic commutant  $U$  of the operator  $G$  reads (see proposition 4 of Ref. [22] )

$$U = \sum_m \sum_{a=1}^{d_m} \sum_{b=1}^{d_m} |u_m, a\rangle u(m, a, b) \langle u_m, b|, \quad u(m, a, b) \in \mathbb{Q}.$$

Moreover  $U$  is unitary if the square matrices  $[u(m, a, b)]$  of dimension  $d_m$  with entries  $u(m, a, b)$  ( $a$  and  $b$  denote row and column indices respectively) belong to the quaternionic unitary group  $U(d_m, \mathbb{Q})$  of dimension  $d_m$ .

Then, the quaternionic group of bi-unitary transformations is given by

$$U(d_1, \mathbb{Q}) \times U(d_2, \mathbb{Q}) \times \dots \times U(d_m, \mathbb{Q}), \quad d_1 + d_2 + \dots + d_m = n. \quad (13)$$

According with definitions (1), if  $h_1$  and  $h_2$  are in generic position, the group of bi-unitary transformations becomes

$$\underbrace{U(1, \mathbb{Q}) \times U(1, \mathbb{Q}) \times \dots \times U(1, \mathbb{Q})}_{n \text{ factors}}. \quad (14)$$

Now, we say that a quaternionic operator  $G$  is cyclic when a vector  $|x_0\rangle$  exists such that the set  $\{|x_0\rangle, G|x_0\rangle, \dots, G^{n-1}|x_0\rangle\}$  spans the whole  $n$ -dimensional right quaternionic vector space  $\mathbb{Q}^n$ , i.e. they are right linearly independent on  $\mathbb{Q}$ , we show that:

**Proposition 2** *Two quaternionic Hermitian forms are in generic relative position if and only if their connecting operator  $G$  is cyclic.*

**Proof** The non singular Hermitian operator  $G$  has a discrete spectrum and is diagonalizable so, when  $h_1$  and  $h_2$  are in generic position,  $G$  admits  $n$  distinct real

12 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

eigenvalues  $\lambda_l$ . Let now  $\{|u_l\rangle\}$  be the eigenvector basis of  $G$  and  $\{\mu_l = c_l + jc'_l\}$  an  $n$ -tuple of non-zero quaternionic numbers. The vector

$$|x_0\rangle = \sum_l |u_l\rangle \mu_l \quad (15)$$

is a cyclic vector for  $G$ . In fact, by applying  $G^m$  to  $|x_0\rangle$  one obtains

$$G^m |x_0\rangle = \sum_l |u_l\rangle \mu_l \lambda_l^m, \quad m = 0, 1, \dots, n-1 \quad (16)$$

and the vectors  $\{G^m |x_0\rangle\}$  are right linearly independent on  $\mathbb{Q}$ . In fact, it is known that the rank of a  $n$ -dimensional quaternionic matrix is  $n$  if and only if its complex counterpart has rank  $2n$ . [23] Then, denoting with  $\Lambda$  the Vandermonde matrix and with  $C = \text{diag}(c_1, \dots, c_n)$  and  $C' = \text{diag}(c'_1, \dots, c'_n)$  two diagonal complex matrices, the complex counterpart of the quaternionic matrix  $M = \Lambda C + j\Lambda C'$  of the components is given by

$$M_c = \begin{pmatrix} \Lambda C & \Lambda C' \\ -(\Lambda C')^* & (\Lambda C)^* \end{pmatrix} = \begin{pmatrix} \Lambda C & \Lambda C' \\ -\Lambda C'^* & \Lambda C^* \end{pmatrix},$$

and by a direct computation one has

$$\det M_c = \left( \prod_k |\mu_k|^2 V^2(\lambda_1, \dots, \lambda_n) \right), \quad (18)$$

where  $V$  denotes the Vandermonde determinant which is different from zero when all the eigenvalues  $\lambda_k$  are distinct. The converse is also true.  $\square$

The equivalence of definitions (1), (2) and (4) is apparent also for finite-dimensional right quaternionic vector spaces.

We conclude this Section noticing that, unlike the complex case, definition (3) is not equivalent to definition (1), (2) and (4) when two Hermitian forms are considered on a right quaternionic vector space. In fact, if for instance definition (3) holds, it is easy to see that the bi-commutant of  $G$  is abelian while its commutant is not commutative according with Eq.(14), hence  $G'' \neq G'$ .

## 5. Alternative compatible algebraic structures

In this Section we will discuss the Heisenberg picture of Quantum bi-Hamiltonian dynamical systems on complex and quaternionic Hilbert spaces limiting ourselves to consider the case of constant tensorial structures.

Looking for alternative quantum Hamiltonian descriptions in the Heisenberg picture is equivalent to search for alternative associative products on the space of observables, with the requirement that the equations of motion define a derivation with respect to the alternative associative product. [9]

We start with some pure algebraic considerations. [9, 17]

Let  $(\mathcal{A}, \cdot)$  be a unital associative algebra with identity  $E$ . A simple way to define a new associative product on  $\mathcal{A}$  is to take an element  $K \in \mathcal{A}$  and to define a new product by

$$A \circ_K B = A \cdot K \cdot B. \quad (1)$$

We denote with  $(\mathcal{A}, \circ_K)$  the new associative algebra; if  $K$  is invertible in  $(\mathcal{A}, \cdot)$ , then  $(\mathcal{A}, \circ_K)$  is also a unital algebra. In this case, the identity in  $(\mathcal{A}, \circ_K)$  is  $E_K = K^{-1}$  and there is an isomorphism  $\varphi$  between the algebras  $(\mathcal{A}, \cdot)$  and  $(\mathcal{A}, \circ_K)$ :

$$\varphi : (\mathcal{A}, \cdot) \rightarrow (\mathcal{A}, \circ_K) : \varphi(A) = \frac{1}{K} \cdot A \quad (2)$$

The two different associative products in  $\mathcal{A}$  allow the introduction of two different Lie products

$$[A, B] = A \cdot B - B \cdot A \quad (3)$$

and

$$[A, B]_K = A \cdot K \cdot B - B \cdot K \cdot A = A \circ_K B - B \circ_K A. \quad (4)$$

From now on we will skip the product symbol  $\cdot$  for the original associative structure.

Let us consider the following Lie subalgebras,  $S \subset \mathcal{A}$

$$S \subset \mathcal{A} : S = \{A : [A, K] = 0\} \quad (5)$$

and  $S_K \subset \mathcal{A}$

$$S_K \subset \mathcal{A} : S_K = \{A : [A, K]_K = 0\}. \quad (6)$$

It is immediate to verify that  $S \subset S_K$ ; furthermore, if  $K$  is invertible (note that the invertibility notion does not depend on the algebra considered) we can exchange the algebras so that it follows  $S = S_K$ .

In order to discuss now the correspondence between quantum bi-Hamiltonian systems in the Schroedinger and in the Heisenberg pictures, we have to think of  $\mathcal{A}$  as an algebra of bounded operators on a Hilbert space,  $\mathcal{H}$  or  $\mathcal{H}^{\mathbb{Q}}$ , provided with two Hermitian structures,  $h_2(\cdot, \cdot)$  and  $h_1(\cdot, \cdot)$ . As said in sections 2 and 3, two compatible (complex or quaternionic) Hermitian structures  $h_2$  and  $h_1$  are related by means of a positive self-adjoint operator  $G = h_1^{-1} \circ h_2$ . Moreover, the presence of two Hermitian structures allows us to define in  $\mathcal{A}$  two involutions, the adjoints, denoted with  $\dagger$  and  $*$  respectively. Then, the following relations hold

$$h_1(Ax, By) = h_1(B^\dagger Ax, y)$$

and

$$h_2(Ax, By) = h_2(B^* Ax, y) = h_1(GAx, By) = h_1(B^\dagger GAx, y). \quad (8)$$

As a consequence of the fact that the adjoints operators  $B^\dagger$  and  $B^*$  belong to the algebra  $\mathcal{A}$  and Eq.(8), it follows that the alternative associative product between operators is given by

$$A \circ_G B = AGB. \quad (9)$$

As said before, we obtain two operator algebras endowed with two different associative products and two corresponding different Lie products. The invertibility of  $G$  leads to the isomorphism  $S = S_G$ . Furthermore, the algebra  $S = S_G$  is

invariant with respect to the two involutions because the operator  $G$  is self-adjoint with respect to both the Hermitian structures and the involutions coincide if we restrict ourselves to  $S = S_G$ .

What happens when  $h_2$  and  $h_1$  are in relative generic position on complex or (right) quaternionic Hilbert spaces?

If  $h_2$  and  $h_1$  are in relative generic position on  $\mathcal{H}$ , the previous analysis shows that  $S$  and  $S_G$  are two abelian isomorphic algebras. Moreover, these two algebras decompose into a direct sum of one-dimensional complex algebras.

On the contrary, if  $h_2$  and  $h_1$  are not in relative generic position on  $\mathcal{H}$ , the previous analysis shows that  $S$  and  $S_G$  are two non abelian isomorphic algebras which decompose into a direct sum of algebras of general complex matrices.

If  $h_2$  and  $h_1$  are in relative generic position on  $\mathcal{H}^{\mathbb{Q}}$ , the previous analysis shows that  $S$  and  $S_G$  are two isomorphic algebras. Moreover, these two algebras decompose into a direct sum of one-dimensional quaternionic algebras.

On the contrary, if  $h_2$  and  $h_1$  are not in relative generic position the previous analysis shows that  $S$  and  $S_G$  are two isomorphic algebras which decompose into a direct sum of algebras of all quaternionic matrices.

So, we have found a correspondence between compatible quantum bi-Hamiltonian systems in the Schroedinger and in the Heisenberg pictures, if the associative product given in Eq.(9) coming from constant Kähler metrics on the Hilbert space is assumed.

It is a simple matter to show that all these alternative associative structures may be added to provide new alternative structures, i.e. they are always compatible. Therefore the problem arises to find out in which conditions we are going to find alternative associative products whose classical limit would give the richness of alternative Poisson structures of the classical situation.

Our belief is that the not necessarily compatible associative products found in [9] might correspond to alternative products not linearly related when realized on the same manifold of states. In the coming section we shall investigate these considerations by means of an example.

Moreover, the next example will show that choosing a non-constant Kähler metric [15] changes the linear structure of the space. In this case, different associative products between operators, all compatible according to ref. [9] have to be introduced, in order to obtain the correspondence between Heisenberg and Schroedinger pictures.

### 5.1. *Example: a two level system*

Here we will discuss the associative product between functions associated with operators, but considering the manifold of states to be  $\mathcal{M} \equiv \mathbb{R}^4$ . A choice of a global chart on  $\mathcal{M}$  allows to write the tensors corresponding to an Hermitian tensor. The metric tensor  $G$  and the symplectic form  $\Omega$  on  $\mathbb{R}^4$  space will be given by

$$G = dP_1 \otimes dP_1 + dQ_1 \otimes dQ_1 + dP_2 \otimes dP_2 + dQ_2 \otimes dQ_2 \quad (10)$$

and

$$\Omega = dP_1 \wedge dQ_1 + dP_2 \wedge dQ_2, \quad (11)$$

so that the complex structure  $J$  is

$$J = dP_1 \otimes \frac{\partial}{\partial Q_1} - dQ_1 \otimes \frac{\partial}{\partial P_1} + dP_2 \otimes \frac{\partial}{\partial Q_2} - dQ_2 \otimes \frac{\partial}{\partial P_2}. \quad (12)$$

We may express them, recalling Eqs. (32), (33) of the previous simple example, in terms of different coordinates  $q_1, p_1, q_2, p_2$ . Then[7]

$$f_A(x) = \frac{1}{2} \langle x, Ax \rangle_{\mathcal{H}} = \frac{1}{2} \langle x, A(G + i\Omega)x \rangle, \quad (13)$$

$$f_{2AB}(x) = [f_A, f_B]_{\mathcal{H}} = (f_A, f_B)_g + i \{f_A, f_B\}_{\omega} \quad (14)$$

where  $x = (q_1 + ip_1, q_2 + ip_2)$ , and

$$(f_A, f_B)_g = \sum_k \left( \frac{\partial f_A}{\partial q_k} \frac{\partial f_B}{\partial q_k} + \frac{\partial f_A}{\partial p_k} \frac{\partial f_B}{\partial p_k} \right)$$

$$\{f_A, f_B\}_{\omega} = \sum_k \left( \frac{\partial f_A}{\partial q_k} \frac{\partial f_B}{\partial p_k} - \frac{\partial f_A}{\partial p_k} \frac{\partial f_B}{\partial q_k} \right).$$

Denoting with  $\sigma_1, \sigma_2$  and  $\sigma_3$  the realification of Pauli matrices on  $\mathbb{R}^4$  space and with  $\sigma_0$  the identity matrix, in the coordinate system  $Q_1, P_1, Q_2, P_2$ , with  $\{Q_1, P_1\} = \{Q_2, P_2\} = 1$ , we have

$$f_{\sigma_0}(x) = \frac{1}{2} [Q_1^2 + P_1^2 + Q_2^2 + P_2^2],$$

$$f_{\sigma_3}(x) = \frac{1}{2} [Q_1^2 + P_1^2 - Q_2^2 - P_2^2],$$

$$f_{2\sigma_0\sigma_3}(x) = \frac{1}{2} \langle x, 2\sigma_0\sigma_3(G + i\Omega)x \rangle = 2f_{\sigma_3}(x) = Q_1^2 + P_1^2 - Q_2^2 - P_2^2$$

and

$$f_{2\sigma_0\sigma_3}(x) = (f_{\sigma_0}, f_{\sigma_3})_g + i \{f_{\sigma_0}, f_{\sigma_3}\}_{\omega}$$

$$= \sum_k \left[ \frac{\partial f_{\sigma_0}}{\partial Q_k} \frac{\partial f_{\sigma_3}}{\partial Q_k} + \frac{\partial f_{\sigma_0}}{\partial P_k} \frac{\partial f_{\sigma_3}}{\partial P_k} + i \left( \frac{\partial f_{\sigma_0}}{\partial Q_k} \frac{\partial f_{\sigma_3}}{\partial P_k} - \frac{\partial f_{\sigma_0}}{\partial P_k} \frac{\partial f_{\sigma_3}}{\partial Q_k} \right) \right]$$

$$= Q_1^2 + P_1^2 - Q_2^2 - P_2^2.$$

On the contrary, it results

$$\frac{1}{2} \langle x, 2\sigma_0\sigma_3(G + i\Omega)x \rangle \neq (f_{\sigma_0}, f_{\sigma_3})_g + i \{f_{\sigma_0}, f_{\sigma_3}\}_{\omega}$$

in the coordinates  $q_1, p_1, q_2, p_2$ , when  $\{Q_1, P_1\} = 1 / [(1 + \lambda(q_1^2 + 3p_1^2))^2 - 4\lambda^2 p_1^2 q_1^2]$  and  $\{Q_2, P_2\} = 1 / [(1 + \lambda(q_2^2 + 3p_2^2))^2 - 4\lambda^2 p_2^2 q_2^2]$ .

However, if we consider standard Poisson structures and Euclidean structures in the  $(q, p)$  variables, the quadratic functions associated with Pauli matrices will define the expected product.

This example shows that we may realize the same abstract algebra in two alternative ways not linearly related. To let them act on the same manifold of states we have to endow this manifold  $\mathcal{M}$  with two alternative linear structures (represented here by the choice of  $(q, p)$  and  $(Q, P)$  variables respectively).

We believe that to realize these alternative algebras as algebras of operators on the same vector space, this vector space must be required to have much larger dimension (in the present case, it should have a dimension greater than four). This situation should be compared with the one of the (non-linear) Riccati equation and its linearization as presented in Ref. [24].

A realization of the Heisenberg algebra in terms of not linearly related creation and annihilation operators has been given in Ref. [17] and considered to describe non-linear oscillators.[25]

### 5.2. A quaternionic example

This example is provided to show that in the quaternionic setting further problems arise with respect to the complex space situation already at the level of “constant” tensor fields.

Let us consider a two level dynamical quantum system in the complex Hilbert space  $\mathcal{H}$  whose dynamics is described by the complex anti-Hermitian time-dependent Hamiltonian ( $\hbar = 1$ )

$$\tilde{H} = 2\Omega_0(t)J_1 + 2\Omega_1(t)J_2 + \omega(t)J_3, \quad (21)$$

where  $\Omega_0(t)$ ,  $\Omega_1(t)$  and  $\omega(t)$  are real valued functions of the time  $t$  and the anti-Hermitian operators  $J_l$  ( $l = 1, 2, 3$ ) obey the usual rules of commutation of the  $su(2)$  algebra:

$$[J_l, J_m] = -\varepsilon_{lmn}J_n.$$

By resorting to the irreducible 2-dimensional representation of the  $J$  operators

$$J_1 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_3 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (23)$$

and putting  $\Omega = \Omega_0 + i\Omega_1$ , we can write the Hamiltonian (21) as a  $2 \times 2$  anti-Hermitian complex matrix :

$$\tilde{H} = i \begin{pmatrix} \frac{\omega(t)}{2} & \Omega^*(t) \\ \Omega(t) & -\frac{\omega(t)}{2} \end{pmatrix}. \quad (24)$$

The set  $\mathfrak{H}$  of anti-Hermitian complex operators obtained by changing the entries in Eq.(24), is of course irreducible in the 2-dimensional Hilbert space  $\mathcal{H}$ , since such is the spinorial representation (23) of the  $J_i$ 's.

From a different point of view, we can interpret the Hamiltonian of Eq.(24) as an anti-Hermitian quaternionic operator in a (right) quaternionic Hilbert space  $\mathcal{H}^{\mathbb{Q}}$ ,



and the dynamics of our quantum system is then described by the Schroedinger equation [10]

$$\frac{d}{dt} \begin{pmatrix} \Psi_+(t) \\ \Psi_-(t) \end{pmatrix} = -i \begin{pmatrix} \frac{\omega(t)}{2} & \Omega^*(t) \\ \Omega(t) & -\frac{\omega(t)}{2} \end{pmatrix} \begin{pmatrix} \Psi_+(t) \\ \Psi_-(t) \end{pmatrix} \quad (25)$$

where  $\Psi_+(t), \Psi_-(t) \in \mathbb{Q}$ .

Roughly speaking,  $\mathcal{H}^{\mathbb{Q}}$  can be obtained from  $\mathcal{H}$  by simply adding to each complex vector  $|v\rangle \in \mathcal{H}$  a term  $|v'\rangle j$ , where  $|v'\rangle \in \mathcal{H}$  and  $j : j^2 = -1$  is a quaternionic unity different from  $i$ ; note that  $\dim \mathcal{H}^{\mathbb{Q}} = \dim \mathcal{H} = 2$ . [18]

The Cayley-Klein (CK) matrix reads [26,27]

$$\begin{pmatrix} \Psi_+(t) \\ \Psi_-(t) \end{pmatrix} = \begin{pmatrix} F^* & L \\ -L^* & F \end{pmatrix} \begin{pmatrix} \Psi_+(0) \\ \Psi_-(0) \end{pmatrix}, \quad (26)$$

where  $F(t)$  and  $L(t)$  are complex functions depending on  $\omega$  and  $\Omega$  in a rather involved way; furthermore  $F(0) = 1$ ,  $L(0) = 0$ , and  $|F|^2 + |L|^2 = 1$  [26].

The CK matrix can be regarded as the matrix representation of the time evolution operator  $U$  associated with the time dependent Hamiltonian (4), and it belongs to a 2-dimensional (complex) unitary representation of the  $SU(2)$  group; by varying  $H$  in  $\mathfrak{H}$ , we correspondingly obtain a set  $\mathfrak{U} = \{U\}$ .

We remark once again that the form of any element  $U \in \mathfrak{U}$  does not depend on the scalar field,  $\mathbb{C}$  or  $\mathbb{Q}$ , adopted. Now, as long as we study the two-level system in  $\mathcal{H}$ , the set  $\mathfrak{U}$  is clearly irreducible, hence, by the corollary of the Schur Lemma, no non-trivial  $G$  exists which commutes with it. Recalling the discussion in sec. 3, we can conclude that the description of the system in  $\mathcal{H}$  is unique.

On the contrary, if we now consider  $\mathfrak{U}$  as a quaternionic group representation acting on  $\mathcal{H}^{\mathbb{Q}}$ , it can be proven that this representation is reducible into the direct sum of two equivalent one-dimensional irreducible quaternionic representations on  $\mathcal{H}^{\mathbb{Q}}$  [28], [29], so that  $\mathfrak{U}$  admits a non-trivial commutant. By a direct computation, the most general quaternionic positive Hermitian matrix  $G$  commuting with any  $U \in \mathfrak{U}$  is

$$G = \begin{pmatrix} a & jz \\ -jz & a \end{pmatrix}, \quad z \in \mathbb{C}, \quad a > |z|. \quad (27)$$

We can conclude that any element  $U \in \mathfrak{U}$  is bi-unitary on  $\mathcal{H}^{\mathbb{Q}}$  with respect to the Hermitian structures  $h_1(\psi, \varphi) = \langle \psi | \varphi \rangle_1$  and  $h_2(\psi, \varphi) = \langle \psi | \varphi \rangle_2$  with  $\langle \psi | \varphi \rangle_2 = \langle \psi | G | \varphi \rangle_1$ :

$$U^\dagger G U = G. \quad (28)$$

Moreover,  $h_1$  and  $h_2$  are in generic position, in fact the eigenvalues of  $G$  are different, as one can prove by solving the eigenvalue problem associated with it [23]:  $\lambda_{1,2} = a \pm |z|$ .

We show now that, according with results in the previous section, the algebras  $\mathfrak{U} \subset S = S_G$  decompose into a direct sum of one-dimensional unimodular

18 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

non-commutative algebras  $\mathbb{Q}$ . In fact by applying the quaternionic unitary transformation  $D = \frac{1}{2} \begin{pmatrix} 1+i & j-k \\ 1-i & j+k \end{pmatrix}$ ,  $D^{-1} = D^\dagger = \frac{1}{2} \begin{pmatrix} 1-i & 1+i \\ k-j & -j-k \end{pmatrix}$  one obtains

$$DU(t)D^\dagger = \begin{pmatrix} F^*(t) - jL^*(t) & 0 \\ 0 & F^*(t) - jL^*(t) \end{pmatrix},$$

$$DJ_1D^\dagger = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad DJ_2D^\dagger = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad DJ_3D^\dagger = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

and

$$DGD^\dagger = \begin{pmatrix} a+|z| & 0 \\ 0 & a-|z| \end{pmatrix}.$$

Finally, making resort to the form (8) of the evolution operator  $U$ , we can also compute the transition probabilities in both the descriptions. Let us for instance assume that the system is in the excited state  $|+\rangle$  at  $t=0$ ; the probability of finding the system in the ground state  $|-\rangle$  at the time  $t$  is given by

$$\mathcal{P}_{+\rightarrow-}(t) = |\langle -|U|+\rangle_1|^2 = |L|^2 \quad (32)$$

according to the first description, and by

$$\mathcal{P}'_{+\rightarrow-}(t) = |\langle -|U|+\rangle_2|^2 = |z|^2|F|^2 + |L|^2 \quad (33)$$

according to the alternative description.

We emphasize in conclusion that the possibility of an alternative description for this model can only occur in QQM, which then appears as a theory intrinsically different from CQM, and not a mere transcription of it. These findings seem to go in the same direction as those found by Kossakowski in describing completely positive maps. [11]

## 6. Conclusions

In this paper, guided by the compatibility condition emerging for Poisson structures when dealing with bi-Hamiltonian completely integrable systems, we have considered the analog problem for quantum systems in the framework of Schroedinger and Heisenberg pictures. In particular, we have concentrated our attention on the equivalence between the two descriptions when nonlinear transformations on the manifold of states are performed. We find that the two pictures are still equivalent when alternative structures are taken into account without changing the linear structure on the manifold of states. To allow for a nontrivial compatibility condition on the associative structure of the observables it seems that we are obliged to perform nonlinear transformations. We have given an example where the mechanism is present, however a reasonable understanding of the equivalence or lack of it between the two pictures (Schroedinger and Heisenberg) when nonlinear transformations are allowed is still missing and further work is required.

## References

- [1] G. Marmo, G. Morandi, A. Simoni and F. Ventriglia, Alternative structures and bi-Hamiltonian systems, *J. Phys. A* **35** (2002) 8393-8406.
- [2] F. Ventriglia, Alternative Hamiltonian Descriptions for Quantum Systems and Non-Hermitian Operators with Real Spectrum, *Mod. Phys. Lett. A* **17** (2002) 1589-1599.
- [3] G. Marmo, A. Simoni and F. Ventriglia, Quantum Systems: Real Spectra and Non-hermitian (Hamiltonian) Operators, *Rep. Math. Phys.* **51** (2003) 275-285.
- [4] G. Marmo, A. Simoni and F. Ventriglia, Quantum Systems and Alternative Unitary Descriptions, *Int. J. Mod. Phys. A* **19** (2004) 2561-2578.
- [5] G. Marmo, G. Sclarici, A. Simoni and F. Ventriglia, Quantum Bi-Hamiltonian systems, alternative Hermitian structures and Bi-Unitary transformations, *Note di Matematica* **23** (2004) 173-181.
- [6] G. Marmo, G. Sclarici, A. Simoni and F. Ventriglia, Classical and Quantum Systems: Alternative Hamiltonian Descriptions, *Theor. Mat. Phys.* in press.
- [7] G. Marmo, G. Sclarici, A. Simoni and F. Ventriglia, The Quantum-Classical Transition: The Fate of the Complex Structure, *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 127-145.
- [8] G. Marmo, G. Sclarici, A. Simoni and F. Ventriglia, Alternative structures and bi-Hamiltonian systems on a Hilbert space, *J. Phys. A* **38** (2005) 3813-3822.
- [9] J. F. Cariñena, J. Grabowski and G. Marmo, Quantum Bi-Hamiltonian Systems, *Int. Jour. Mod. Phys.* **30** (2000) 4797-4810.
- [10] S.L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford University, New York, 1995).
- [11] A. Kossakowski, Remarks on Positive Maps on Finite Dimensional Simple Jordan Algebras, *Rep. Math. Phys.* **46** (2000) 393-397.
- [12] V.I. Man'ko, G. Marmo, E.C.G. Sudarshan and F. Zaccaria, Differential geometry of density states, to appear on *Rep. Math. Phys.*
- [13] R. de Ritis, G. Marmo, B. Preziosi, A new look at relativity transformations, *Gen. Rel. Grav.* **31** (1999) 1501.
- [14] G. Morandi, C. Ferrario and G. Lo Vecchio, G. Marmo and C. Rubano, The inverse problem in the calculus of variations and the geometry of the tangent bundle, *Phys. Rep.* **188**(1990) 147-284.
- [15] B.A. Dubrovin, G. Marmo and A. Simoni, Alternative Hamiltonian Description for Quantum Systems, *Mod. Phys. Lett. A* **15** (1990) 1229-1234.
- [16] G. Marmo, A. Simoni and F. Ventriglia, Bi-Hamiltonian quantum system and Weyl quantization, *Rep. Math. Phys.* **48** (2001) 149-157.
- [17] V.I. Man'ko, G. Marmo and F. Zaccaria, E.C.G. Sudarshan, Wigner's Problem and Alternative Commutation Relations for Quantum Mechanics, *Int. J. Mod. Phys. B* **11** (1997) 1281-1296.
- [18] C. S. Sharma, Complex structure on a real Hilbert space and symplectic structure on a complex Hilbert space, *J. Math. Phys.* **29** (1988) 1069-1078.
- [19] G. Gaeta and P. Morando, Hyper-Hamiltonian dynamics, *J. Phys. A* **35** (2002) 3925-3943.
- [20] V. Bargmann, Note on Wigner's Theorem on Symmetry Operators, *J. Math. Phys.* **5** (1964) 862-868.
- [21] L. P. Horwitz and L. C. Biedenharn, Quaternion Quantum Mechanics: Second Quantization and Gauge Fields, *Ann. Phys.* **157** (1984) 432-488.
- [22] G. Sclarici, Pseudoanti-Hermitian operators in quaternionic quantum mechanics, *J. Phys. A* **35** (2002) 7493-7505.
- [23] F. Zhang, Quaternions and Matrices of Quaternions, *Lin. Alg. Appl.* **251**

20 *G Marmo, G Sclarici, A Simoni, F Ventriglia*

- (1997) 21-57.
- [24] J.F. Cariñena, J. Grabowski and G. Marmo, *Lie-Scheffers Systems: A geometric Approach* (Bibliopolis, Napoli, 2000).
  - [25] V.I. Man'ko, G. Marmo and F. Zaccaria, E.C.G. Sudarshan, f-oscillators and nonlinear coherent states, *Phys. Scripta* 55 (1997) 528-541.
  - [26] G. Dattoli and A. Torre, Cayley-Klein parameters and evolution of two- and three-level systems and squeezed states, *J. Math. Phys.* 31 (1990) 236-240.
  - [27] A. Blasi, G. Sclarici and L. Solombrino, Alternative descriptions in quaternionic quantum mechanics, *J. Math. Phys.* 46 (2005) 42104-42107.
  - [28] D. Finkelstein, J. M. Jauch, S. Sciminovich and D. Speiser, Quaternionic Representations of Compact Groups, *J. Math. Phys.* 4 (1963) 136-140.
  - [29] G. Sclarici and L. Solombrino, Quaternionic representations of magnetic groups, *J. Math. Phys.* 38 (1997) 1147-1160.